Algebraic Classification of Weyl Anomalies in Arbitrary Dimensions

Nicolas Boulanger
Université de Mons-Hainaut, Académie Wallonie-Bruxelles,
Mécanique et Gravitation, Avenue du Champ de Mars 6, B-7000 Mons, Belgium

Conformally invariant massless field systems involving only dimensionless parameters are known to describe particle physics at very high energy. In the presence of an external gravitational field, the conformal symmetry may generalize to Weyl invariance. However, the latter symmetry no longer survives after quantization: A Weyl anomaly appears. In this Letter, a purely algebraic understanding of the universal structure of the Weyl anomalies is presented. The results hold in arbitrary dimensions and independently of any regularization scheme.

arXiv:0706.0340v1 [hep-th] 3 Jun 2007

At very high energies, like e.g. in the early Universe, all the particles can be considered as massless, and renormalized matter models are invariant under the conformal group. Since the Weyl transformations are the generalization to curved space of conformal transformations in flat space, there are good reasons to anticipate the Weyl symmetry as a symmetry of a fundamental theory incorporating gravity [1].

On the other hand, symmetries may be broken at the quantum level: Anomalies then appear. The cancellation of anomalies puts severe constraints on the physical content of a theory, as is the case with the Standard Model (for a review on anomalies in quantum field theory, see e.g. [2]). In the case of (super)string theory, the critical dimensions correspond to the absence of the two-dimensional Weyl anomaly [3].

The Weyl (or conformal, or trace) anomalies have been discovered about 30 years ago [4, 5] and still occupy a central position in theoretical physics, partly because of their important rôles within the AdS/CFT correspondence and their many applications in cosmology, particle physics, higher-dimensional conformal field theory, supergravity and strings. The body of work devoted to this subject is, therefore, considerable. A very non-exhaustive list of references can be found, e.g., in [6, 7, 8, 9]

The central equations which determine the candidate anomalies in quantum field theory are the Wess-Zumino (WZ) consistency conditions [10]. By using these conditions, the general structure of all the know anomalies *except the Weyl ones* has been determined by purely algebraic methods featuring descent equations à la Stora-Zumino [11, 12]. Such algebraic treatments are crucial since they are independent of any regularization scheme and very general. The algebraic analysis of anomalies can best be performed within the Becchi-Rouet-Stora-Tyutin (BRST [13]) formulation.

The BRST formulation for the determination of the Weyl anomalies was initiated in the pioneering works [14, 15], with explicit results up to spacetime dimension $n = 6$ and the general structure guessed in arbitrary even dimension. The authors of [14, 15] found that the Weyl anomalies comprise (i) the integral over spacetime of the Weyl scaling parameter times the Euler density of the manifold, plus (ii) terms that are given by (the integral of) the Weyl parameter times strictly Weyl-invariant scalar densities. Some of the terms from (ii) can trivially be obtained from contractions of products of the conformally invariant Weyl tensor, while the others are more complicated and involve covariant derivatives of the Riemann tensor. It was also mentioned in [15] that an algebraic analysis of the Weyl anomalies, similar to the Stora-Zumino treatment of the non-Abelian chiral anomaly in Yang-Mills theory, was unlikely to exist.

Somewhat later, by using dimensional regularization, the authors of [16] confirmed the structure of the Weyl anomalies found in [14, 15] and extended the results to arbitrary (even) dimensions. The Euler term from class (i) was called "type-A Weyl anomaly", while the terms of (ii) were called "type-B anomalies". Very interestingly, they discovered a similitude between the type-A Weyl anomaly and the non-Abelian chiral anomaly. Accordingly, they hinted at the existence of an algebraic treatment for the Weyl anomaly, featuring descent equations.

In this Letter, we provide for the Weyl anomalies the general, purely algebraic understanding à la Stora-Zumino that all the other known anomalies in quantum field theory enjoy, thereby filling a gap in the literature.

The Weyl anomaly being a local functional, *i.e.* the integral over the n-dimensional spacetime manifold $\mathscr{M}_n$ of a local n-form a^n_1 at ghost number unity, $gh(a^n_1) = 1$ (cf. [17]), the WZ consistency conditions for the Weyl anomalies [14, 15] can be written in terms of local forms:

$$\begin{cases} s_{_W} a^n_1 + d\, b^{n-1}_2 = \ 0\ , \\ s_{_D} a^n_1 + d\, c^{n-1}_2 = \ 0\ , \end{cases} \tag{1}$$

$$\begin{cases} a^n_1 \neq \ s_{_W} p^n_0 + d\, f^{n-1}_1 \\ \forall\ \ p^n_0\ \ \text{s.t.}\ \ s_{_D} p^n_0 + d\, h^{n-1}_1 = 0\ . \end{cases} \tag{2}$$

The BRST differentials $s_{_W}$ and $s_{_D}$ implement the Weyl transformations and the diffeomorphisms, respectively, whereas d denotes the exterior total derivative. Together with the invertible spacetime metric $g_{\mu\nu}$, the other fields of the problem are the Weyl ghost ω and the diffeomorphisms ghosts ξ^μ , $gh(\xi^\mu) = gh(\omega) = 1$. The BRST transformations on the fields $\Phi^A = \{g_{\mu\nu}, \omega, \xi^\mu\}$ read

$$s_{_D} g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}\ ,\ s_{_W} g_{\mu\nu} = 2\omega g_{\mu\nu}$$
$$s_{_D} \xi^\mu = \xi^\rho \partial_\rho \xi^\mu\ , \quad s_{_D} \omega = \xi^\rho \partial_\rho \omega\ , \quad s_{_W} \xi^\mu = 0 = s_{_W} \omega\ .$$

It should be understood, throughout this Letter, that the space in which BRST-cohomologies are to be computed is the space of local p-forms b^p, that is, the (jet) space of spacetime p-forms that depend on the fields Φ^A and their derivatives up to some finite (but otherwise unspecified) order, which one denotes [17] by $b^p = \frac{1}{p!}\, dx^{\mu_1} \dots dx^{\mu_p}\, b_{\mu_1 \dots \mu_p}(x, [\Phi^A])$.

One unites the differentials $s = s_{_W} + s_{_D}$ and d into a single differential $\tilde{s} = s + d$, therefore working with local total forms. The latter are, by definition, formal sums of local forms with different form degrees and ghost numbers, $\alpha = \sum_{p=0}^{n} a^p_{G-p}$, the total degree G being simply the sum of the form degree and the ghost number.

Powerful techniques for the computation of local BRST cohomologies in top form degree are exposed in [18] and allow one to consider local total forms depending only on a subset $\mathscr{W}$ of the set of local total forms, such that $\tilde{s}\mathscr{W} \subset \mathscr{W}$. For the general class of theories studied here, the corresponding space $\mathscr{W}$ was obtained in [19].

Accordingly, denoting $\tilde{s}_{_W} = s_{_W} + d$ and similarly for $s_{_D}$, the problem (1)–(2) amounts to determining the $\tilde{s}_{_D}$-invariant $(n+1)$-local total forms $\alpha(\mathscr{W})$ satisfying

$$\tilde{s}_{_W} \alpha(\mathscr{W}) = 0\,, \quad \alpha(\mathscr{W}) \neq \tilde{s}_{_W} \zeta(\mathscr{W}) + constant\,, \quad (3)$$

where $\zeta(\mathscr{W})$ must be $\tilde{s}_{_D}$-invariant.

Thanks to very general results explained in [18], we know that the solution of (3) will take the form

$$\alpha(\mathscr{W}) = 2\omega\, \tilde{C}^{N_1} \dots \tilde{C}^{N_n}\, a_{N_1 \dots N_n}(\mathscr{T})\,. \quad (4)$$

The space $\mathscr{T}$ is generated by [19] the (invertible) metric $g_{\mu\nu}$ together with the W-tensors $\{W_{\Omega_i}\}$, $i \in \mathbb{N}$, whose precise form will not be needed here. For the purposes of the present Letter, it suffices to know that they contain the conformally invariant Weyl tensor $W^\mu{}_{\nu\rho\sigma}$ and its first covariant derivative $\nabla_\tau W^\mu{}_{\nu\rho\sigma}$. The symbol ∇ denotes the usual torsion-free metric-compatible covariant differential associated with the Christoffel symbols $\Gamma^\mu{}_{\nu\rho}$. The Ricci tensor is $\mathscr{R}_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}$, where $R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} + \dots$ is the Riemann tensor. The scalar curvature is given by $\mathscr{R} = g^{\alpha\beta}\mathscr{R}_{\alpha\beta}$. Then, one can write the Weyl tensor as $W^\mu{}_{\nu\rho\sigma} = R^\mu{}_{\nu\rho\sigma} - 2\left(\delta^\mu_{[\rho} K_{\sigma]\nu} - g_{\nu[\rho} K_{\sigma]}{}^\mu\right)$, where the tensor $K_{\mu\nu} = \frac{1}{n-2}\left(\mathscr{R}_{\mu\nu} - \frac{1}{2(n-1)}\, g_{\mu\nu}\mathscr{R}\right)$ plays a key rôle in the classification of the Weyl anomalies, as we will see. Square brackets denote strength-one complete antisymmetrization. We also need to recall the definition of the Cotton tensor: $C_{\alpha\rho\sigma} = 2\,\nabla_{[\sigma} K_{\rho]\alpha}$.

The so-called generalized connections $\tilde{C}^N$ in (4) are obtained from [19] after setting the diffeomorphisms ghosts ξ^μ to zero. They read explicitly

$$\{\tilde{C}^N\} = \{2\omega\,, dx^\nu\,, \tilde{C}^\mu{}_\nu\,, \tilde{\omega}_\alpha\}\,,$$
$$\tilde{C}^\mu{}_\nu = \Gamma^\mu{}_{\nu\rho}\, dx^\rho\,, \quad \tilde{\omega}_\alpha = \omega_\alpha - K_{\alpha\rho}\, dx^\rho\,, \quad \omega_\alpha = \partial_\alpha \omega\,.$$

As anticipated, the generalized connections $\tilde{\omega}_\alpha$ play a crucial rôle in the classification of the Weyl anomalies. They decompose into a ghost part ω_α and a "connection" one-form component $\mathscr{A}_\alpha = -K_{\alpha\rho}\, dx^\rho$. The decomposition of $\tilde{s}_{_W}$ with respect to the $\tilde{\omega}_\alpha$-degree is at the core of the descent giving the type-A Weyl anomalies. The differential $\tilde{s}_{_W}$ decomposes into a part noted $\tilde{s}_\flat$ which lowers the $\tilde{\omega}_\alpha$-degree by one unit, a part $\tilde{s}_\natural$ which does not change the $\tilde{\omega}_\alpha$-degree and a part noted $\tilde{s}_\sharp$ which raises the $\tilde{\omega}_\alpha$-degree by one unit.

Before displaying the action of $\tilde{s}_{_W}$ on $\mathscr{W}$, we need to introduce some further objects: (i) the two-forms $W^\mu{}_\nu = \frac{1}{2}\, dx^\rho dx^\sigma\, W^\mu{}_{\nu\rho\sigma}$, $R^{\mu\nu} = \frac{1}{2}\, dx^\rho dx^\sigma R^{\mu\nu}{}_{\rho\sigma}$ and $C_\alpha = \frac{1}{2}\, dx^\rho dx^\sigma\, C_{\alpha\rho\sigma}$, (ii) the symbol $\mathscr{P}^{\mu\alpha}_{\rho\,\nu} = (-g^{\mu\alpha} g_{\rho\nu} + \delta^\mu_\rho \delta^\alpha_\nu + \delta^\mu_\nu \delta^\alpha_\rho)$, (iii) the generators $\Delta^\mu{}_\nu$ of $GL(n)$-transformations of world indices acting on a type-$(1,1)$ tensor T^β_α as $\Delta^\mu{}_\nu T^\beta_\alpha = \delta^\mu_\alpha T^\beta_\nu - \delta^\beta_\nu T^\mu_\alpha$, and (iv) the Weyl-covariant operator $\mathscr{D}_\mu = \nabla_\mu + K_{\mu\alpha}\mathbf{\Gamma}^\alpha$. The definition of the generators $\mathbf{\Gamma}^\alpha$ is not needed here and can be found in [19]. These generators enter the formula for the Weyl transformation of the W-tensors: $s_{_W} W_{\Omega_i} = \omega_\alpha \mathbf{\Gamma}^\alpha W_{\Omega_i}$. Both the Cotton two-form C_α and the generalized connection $\tilde{\omega}_\alpha$ take their values along the generators $\mathbf{\Gamma}^\alpha$, $\mathbf{C} = C_\alpha \mathbf{\Gamma}^\alpha$ and $\tilde{\omega} = \omega_\alpha \mathbf{\Gamma}^\alpha$. The Weyl two-form takes its values along the $GL(n)$ generators: $\mathbf{W} = W^\mu{}_\nu \Delta^\nu{}_\mu$. Finally, we denote by $\varepsilon^{\mu_1 \cdots \mu_n}$ the totally antisymmetric Levi-Civita weight-1 density.

Then, the action of $\tilde{s}_{_W}$ on $\mathscr{W}$ is given in Table I, following a decomposition with respect to the $\tilde{\omega}_\alpha$-degree.

	$\tilde{s}_\flat$	$\tilde{s}_\natural$	$\tilde{s}_\sharp$
$\tilde{\omega}_\alpha$	C_α	$\tilde{C}^\beta{}_\alpha\, \tilde{\omega}_\beta$	0
ω	0	0	$dx^\mu \tilde{\omega}_\mu$
W_{Ω_i}	0	$\tilde{C}^\mu{}_\nu \Delta^\nu{}_\mu W_{\Omega_i} + dx^\mu \mathscr{D}_\mu W_{\Omega_i}$	$\tilde{\omega}_\alpha \mathbf{\Gamma}^\alpha W_{\Omega_i}$
$g_{\alpha\beta}$	0	$\tilde{C}^\mu{}_\nu \Delta^\nu{}_\mu\, g_{\alpha\beta} + 2\omega\, g_{\alpha\beta}$	0
$\tilde{C}^\mu{}_\nu$	0	$W^\mu{}_\nu - \tilde{C}^\mu{}_\alpha \tilde{C}^\alpha{}_\nu$	$\mathscr{P}^{\mu\alpha}_{\rho\,\nu}\, \tilde{\omega}_\alpha\, dx^\rho$

TABLE I: Action of $\tilde{s}_{_W}$, decomposed w.r.t the $\tilde{\omega}_\alpha$-degree

We can now state the following two theorems, the central results reported in this Letter:

Theorem 1: Let $\psi_{\mu_1 \dots \mu_{2p}}$ be the local total form

$$\psi_{\mu_1 \dots \mu_{2p}} = \frac{\omega}{\sqrt{-g}}\, \varepsilon^{\alpha_1 \dots \alpha_r}{}_{\nu_1 \dots \nu_r \mu_1 \dots \mu_{2p}} \times \tilde{\omega}_{\alpha_1} \dots \tilde{\omega}_{\alpha_r}\, dx^{\nu_1} \dots dx^{\nu_r}\,,$$
$$p = m - r\,, \quad m = n/2\,, \quad 0 \leqslant r \leqslant m$$

and $W^{\mu\nu}$ the tensor-valued two-form $W^{\mu\nu} = W^\mu{}_\rho\, g^{\rho\nu}$. Then, the local total forms $\Phi_r^{[n-r]}$ $(0 \leqslant r \leqslant m)$

$$\Phi_r^{[n-r]} = \frac{(-1)^p}{2^p}\, \frac{m!}{r!\, p!}\, \psi_{\mu_1 \dots \mu_{2p}}\, W^{\mu_1 \mu_2} \dots W^{\mu_{2p-1} \mu_{2p}}$$

obey the descent of equations

$$\begin{cases} \tilde{s}_\flat \Phi_r^{[n-r]} + \tilde{s}_\natural \Phi_{r-1}^{[n-r+1]} = 0 \quad , \\ \tilde{s}_\sharp \Phi_r^{[n-r]} = 0 \quad , \quad (1 \leqslant r \leqslant m) \end{cases}$$
$$\tilde{s}_\flat \Phi_1^{[n-1]} = 0 = \tilde{s}_{_W} \Phi_0^{[n]}\,,$$

so that the following relations hold: $\tilde{s}_W\alpha = 0 = \tilde{s}_W\beta$, with $\alpha = \sum_{r=1}^{m} \Phi_r^{[n-r]}$ and $\beta = \Phi_0^{[n]}$.

<u>*Theorem 2:*</u> (A) The top form-degree component a_1^n of α (cf. Theorem 1) satisfies the WZ consistency conditions for the Weyl anomalies. The WZ conditions for a_1^n give rise to a non-trivial descent and a_1^n is the *unique* anomaly with such a property, up to the addition of trivial terms and anomalies satisfying a trivial descent.

(B) The top form-degree component e_1^n of $(\alpha+\beta)$ is proportional to the Euler density of the manifold $\mathscr{M}_n$:

$$e_1^n \;=\; \frac{(-1)^m}{2^m}\,\sqrt{-g}\,\omega\,(R^{\mu_1\nu_1}\dots R^{\mu_m\nu_m})\,\varepsilon_{\mu_1\nu_1\dots\mu_m\nu_m}\;.$$

Clearly, the anomaly $\beta = \Phi_0^{[n]}$ satisfies a trivial descent since it is given by a contraction of a product of Weyl tensors (m of them in dimension $n=2m$).

<u>*Proofs:*</u> The existence part of the non-trivial descent problem for the Weyl anomalies is given in Theorem 1 and part (B) of Theorem 2. It is proved by direct computation. Only part (A) of Theorem 2, the uniqueness part of the problem, is not straightforward. The detailed proof is given elsewhere [20]. It follows lines of reasonings as in e.g. [17, 21, 22] and uses general results given in [23]. The essential point is to determine the most general expression at the bottom of the non-trivial descents associated with the Weyl anomalies. (The anomalies that satisfy the trivial descent $s_W a_1^n = 0$ are the type-B Weyl anomalies [16]; they can be classified along the lines of [19, 24].) It turns out [20] that the most general element at the bottom of these descents is the component of $\Phi_m^{[m]}$ with maximal ghost number $m+1$.

We now illustrate our two theorems with the descents corresponding to $n = 2$, 4 and 6. The general case can readily be understood from these three examples.

The case $n = 2$ is a bit special. Although $K_{\mu\nu}$ is not determined ($\sim \frac{0}{0}$), its trace $K_\rho{}^\rho = \mathscr{R}/(2n-2)$ is well-defined. Theorem 1 gives $\Phi_0^{[2]} = 0$ and $\alpha = \Phi_1^{[1]} = \frac{\omega}{\sqrt{-g}}\,\varepsilon^{\mu\rho}g_{\rho\nu}\tilde{\omega}_\mu dx^\nu = \omega\sqrt{-g}\,\varepsilon_{\rho\nu}g^{\rho\mu}\tilde{\omega}_\mu dx^\nu$. Taking the top form degree of α, we find $a_1^2 = \frac{\omega}{2}\sqrt{-g}\,\mathscr{R}\,d^2x$, the well-known result for the Weyl anomaly in two dimensions.

Next, using Theorem 1 in the case $n=4$ gives

$$\begin{aligned}
\Phi_0^{[4]} &= \frac{\omega}{4}\sqrt{-g}\,\varepsilon_{\mu_1\dots\mu_4}\,W^{\mu_1\mu_2}W^{\mu_3\mu_4}\,,\\
\Phi_1^{[3]} &= -\,\omega\sqrt{-g}\,\varepsilon^{\alpha}{}_{\nu\rho\sigma}\,\tilde{\omega}_\alpha\,dx^\nu\,W^{\rho\sigma}\,,\\
\Phi_2^{[2]} &= \omega\sqrt{-g}\,\varepsilon^{\alpha\beta}{}_{\rho\sigma}\,\tilde{\omega}_\alpha\tilde{\omega}_\beta\,dx^\rho dx^\sigma\,.
\end{aligned}$$

The top form-degree component of $(\alpha+\beta)$ is e_1^4:

$$e_1^4 \;=\; \frac{\omega}{4}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}(W^{\mu\nu} - 2\,\mathscr{A}^\mu dx^\nu)(W^{\rho\sigma} - 2\,\mathscr{A}^\rho dx^\sigma)$$

which obviously reproduces the expression for the Euler term of Theorem 2 because of the following identities:

$$R^{\mu\nu} = W^{\mu\nu} - 2\,\mathscr{A}^{[\mu}dx^{\nu]}\;,\quad \mathscr{A}^\mu = -g^{\mu\nu}K_{\nu\rho}dx^\rho\;. \quad (5)$$

The descent for $n=4$ thus reads

$$\left\{\begin{aligned}
s_W\,e_1^4 + d\,b_2^3 &= 0\;,\\
s_W\,b_2^3 + d\,b_3^2 &= 0\;,\\
s_W\,b_3^2 &= 0\;,\qquad \text{with}
\end{aligned}\right.$$

$$\begin{aligned}
b_2^3 &= -2\,\omega\sqrt{-g}\,\varepsilon^{\alpha}{}_{\nu\rho\sigma}\,\omega_\alpha\,K^\nu{}_\mu\,dx^\mu dx^\rho dx^\sigma\,,\\
b_3^2 &= \omega\sqrt{-g}\,\varepsilon^{\alpha\beta}{}_{\rho\sigma}\,\omega_\alpha\omega_\beta\,dx^\rho dx^\sigma\,.
\end{aligned}$$

Finally, in dimension 6, Theorem 1 and Theorem 2 give (a representative of) the unique Weyl anomaly satisfying a non-trivial descent of equations:

$$e_1^6 \;=\; \frac{-\omega}{8}\sqrt{-g}\,\varepsilon_{\mu_1\dots\mu_6}R^{\mu_1\mu_2}R^{\mu_3\mu_4}R^{\mu_5\mu_6}\,. \quad (6)$$

The elements of the corresponding descent are obtained, as before, *via* the $\Phi_r^{[n-r]}$'s of Theorem 1:

$$\begin{aligned}
\beta = \Phi_0^{[6]} &= \frac{-\omega}{8}\sqrt{-g}\,\varepsilon_{\mu_1\dots\mu_6}\,W^{\mu_1\mu_2}W^{\mu_3\mu_4}W^{\mu_5\mu_6}\,,\\
\Phi_1^{[5]} &= \frac{3\,\omega}{4}\sqrt{-g}\,\varepsilon^{\alpha}{}_{\nu\mu_1\dots\mu_4}\,\tilde{\omega}_\alpha\,dx^\nu\,W^{\mu_1\mu_2}W^{\mu_3\mu_4}\,,\\
\Phi_2^{[4]} &= \frac{-3\,\omega}{2}\sqrt{-g}\,\varepsilon^{\alpha\beta}{}_{\mu\nu\rho\sigma}\,\tilde{\omega}_\alpha\tilde{\omega}_\beta\,dx^\mu dx^\nu\,W^{\rho\sigma}\,,\\
\Phi_3^{[3]} &= \omega\sqrt{-g}\,\varepsilon^{\alpha\beta\gamma}{}_{\mu\nu\rho}\,\tilde{\omega}_\alpha\tilde{\omega}_\beta\tilde{\omega}_\gamma\,dx^\mu dx^\nu dx^\rho\,.
\end{aligned}$$

Extracting from $\alpha = \Phi_1^{[5]} + \Phi_2^{[4]} + \Phi_3^{[3]}$ its top form-degree component amounts to selecting everywhere the contribution $\mathscr{A}_\mu$ of $\tilde{\omega}_\mu = \omega_\mu + \mathscr{A}_\mu$. As a consequence, the top form-degree component of $(\alpha+\beta)$ reproduces the expression (6), making use of the identities (5). On the other hand, extracting the different ghost-number components of α provides us with the elements b_2^5, b_3^4 and b_4^3 of the descent for e_1^6:

$$\left\{\begin{aligned}
s_W\,e_1^4 + d\,b_2^5 &= 0\;,\\
s_W\,b_2^5 + d\,b_3^4 &= 0\;,\\
s_W\,b_3^4 + d\,b_4^3 &= 0\;,\\
s_W\,b_4^3 &= 0\;.
\end{aligned}\right.$$

Without the addition of the type-B anomaly β, the top form-degree component a_1^6 of α, taken alone, gives

$$\begin{aligned}
a_1^6 \;=\; \frac{-3\,\omega}{8}\sqrt{-g}\,\varepsilon_{\mu_1\dots\mu_6}\Big[&(-2\,\mathscr{A}^{\mu_1}dx^{\mu_2})W^{\mu_3\mu_4}W^{\mu_5\mu_6}\\
&+(-2\,\mathscr{A}^{\mu_1}dx^{\mu_2})(-2\,\mathscr{A}^{\mu_3}dx^{\mu_4})W^{\mu_5\mu_6}\\
&+(-2\,\mathscr{A}^{\mu_1}dx^{\mu_2})(-2\,\mathscr{A}^{\mu_3}dx^{\mu_4})(-2\,\mathscr{A}^{\mu_5}dx^{\mu_6})\Big]\,.
\end{aligned}$$

As we have shown, adding $\beta = \Phi_0^{[n]}$ to a_1^n somehow "covariantizes" the latter, producing the Euler term e_1^n. The Weyl anomaly a_1^n is reminiscent of the consistent non-Abelian chiral anomaly. However, note that the descent for a_1^n stops at form-degree $\frac{n}{2} > 0$. Amusingly, the Euler form e_1^n looks like the non-Abelian singlet anomaly. The "trace over the internal indices" is taken with the Levi-Civita density.

<u>*Conclusions :*</u> The universal structure of the Weyl anomalies is established in a purely algebraic manner, independently of any regularization scheme and in arbitrary dimensions. In particular, we do not resort to dimensional analysis. The type-A Weyl anomaly of [16] is the counterpart of the consistent non-Abelian chiral anomaly, in that it is the *unique* Weyl anomaly satisfying a non-trivial descent of equations. This solves a long-standing problem and answers a question originally due to Deser and Schwimmer [16]. Since the Weyl anomalies associated with a trivial descent can be systematically built and classified as in [19, 24], our analysis completes a general, purely algebraic classification of the Weyl anomalies in arbitrary spacetime dimensions.

This work was supported by the Fonds de la Recherche Scientifique, FNRS (Belgium). We thank G. Barnich for many useful discussions and M. Henneaux for having suggested the project. We thank H. Osborn and Ph. Spindel for their comments and encouragements.

[1] E. S. Fradkin and A. A. Tseytlin, Phys. Rept. **119** (1985) 233.
[2] R. A. Bertlmann, "Anomalies in quantum field theory," Oxford, UK: Clarendon (1996).
[3] A. M. Polyakov, Phys. Lett. B **103** (1981) 207; Phys. Lett. B **103** (1981) 211.
[4] D. M. Capper and M. J. Duff, Nuovo Cim. A **23** (1974) 173.
[5] S. Deser, M. J. Duff and C. J. Isham, Nucl. Phys. B **111** (1976) 45.
[6] N. D. Birrell and P. C. W. Davies, "Quantum Fields In Curved Space," Cambridge University Press, UK (1982).
[7] M. J. Duff, Class. Quantum Grav. **11** (1994) 1387 [arXiv:hep-th/9308075].
[8] H. Osborn and A. C. Petkou, Annals Phys. **231** (1994) 311 [arXiv:hep-th/9307010];
J. Erdmenger and H. Osborn, Nucl. Phys. B **483** (1997) 431 [arXiv:hep-th/9605009].
[9] F. Bastianelli and P. van Nieuwenhuizen, "Path integrals and anomalies in curved space," Cambridge University Press, UK (2006).
[10] J. Wess and B. Zumino, Phys. Lett. B **37** (1971) 95.
[11] R. Stora, "Continuum gauge theories," in *New developments in quantum field theory and statistical mechanics.* 1976 Cargèse Lectures, eds. M. Lévy and P. Mitter, Plenum Press, New York; "Algebraic structure and topological origin of anomalies," in *Recent progress in gauge theories.* 1983 Cargèse Lectures, ed. H. Lehmann, NATO ASI series, Plenum Press, New York.
[12] B. Zumino, "Chiral Anomalies And Differential Geometry" in *Relativity, groups and topology II.* Les Houches Lectures, August 1983, eds. B. S. DeWitt and R. Stora, North Holland, Amsterdam.
[13] C. Becchi, A. Rouet and R. Stora, Commun. Math. Phys. **42** (1975) 127; Annals Phys. **98** (1976) 287; I. V. Tyutin, *Gauge Invariance In Field Theory And Statistical Physics In Operator Formalism*, LEBEDEV-75-39; J. Zinn-Justin, *Renormalisation of gauge theories*, Lecture notes in Physics n° 37, Springer, Berlin (1975).
[14] L. Bonora, P. Cotta-Ramusino and C. Reina, Phys. Lett. B **126** (1983) 305.
[15] L. Bonora, P. Pasti and M. Bregola, Class. Quantum Grav. **3** (1986) 635.
[16] S. Deser and A. Schwimmer, Phys. Lett. B **309** (1993) 279 [arXiv:hep-th/9302047]; see also S. Deser, Helv. Phys. Acta **69** (1996) 570 [arXiv:hep-th/9609138].
[17] G. Barnich, F. Brandt and M. Henneaux, Phys. Rept. **338** (2000) 439 [arXiv:hep-th/0002245].
[18] F. Brandt, Commun. Math. Phys. **190** (1997) 459 [arXiv:hep-th/9604025].
[19] N. Boulanger, J. Math. Phys. **46** (2005) 053508 [arXiv:hep-th/0412314].
[20] N. Boulanger, arXiv:0704.2472 [hep-th].
[21] G. Barnich, F. Brandt and M. Henneaux, Nucl. Phys. B **455** (1995) 357 [arXiv:hep-th/9505173].
[22] F. Brandt, Annals Phys. **259** (1997) 253 [arXiv:hep-th/9609192].
[23] A. Barkallil, G. Barnich and C. Schomblond, J. Math. Phys. **43** (2002) 5987 [arXiv:math-ph/0205047].
[24] N. Boulanger and J. Erdmenger, Class. Quantum Grav. **21** (2004) 4305 [arXiv:hep-th/0405228].